Armed Services Technical Information Agency

Because of our limited supply, you are requested to return this copy WHEN IT HAS SERVED YOUR PURPOSE so that it may be made available to other requesters. Your cooperation will be appreciated.

AD 37460

Reproduced by

DOCUMENT SERVICE CENTER

KNOTT BUILDING, DAYTON, 2, OHIO

UNCLASSIFIED

WORKING PAPER. W-53-35

U. S. NAVAL ORDNANCE PLANT
Indianapolis, Indiana
RESEARCH AND TEST DEPARTMENT

LAGRANGIAN FORMULAE

by

Kaj L. Nielsen

19

PRELIMINARY DATA

This is an informal report and is transmitted for information only. The data presented are tentative and subject to later revision.

21 Dec 1953

9ND-P-1487
Navy—DPPO 9ND, Great Lakes, Ill

U. S. NAVAL ORDNANCE PLANT
Indianapolis 18, Indiana
Captain Mell A. Peterson, USN
Commanding

LAGRANGIAN FORMULAE

by

Kaj L. Nielsen

Research and Test Department

21 Dec 1953

FOREWORD

Applied mathematicians are continuously concerned with numerical methods to obtain answers to practical problems. Many of these numerical methods revolve around a well known formula for polynomial representation of a function due to Lagrange. It is the purpose of this paper to present this formula and apply it to many numerical methods and to collect together appropriate tables which will reduce the amount of labor necessary to obtain numerical answers.

Although this formula should only be applied to functions which may be approximated by a polynomial, it is recalled that Weierstrass Theorems shows this property to hold for a large class of functions. It is, of course, true that the degree of the polynomial is not specified and if a function is fitted by a polynomial to a certain specified degree, the accuracy of the fit should be checked.

The collection of derivation of these formulae and tables form a part of a study of numerical methods and the work was performed under Allotment 45007, Job Order 2693.

K. L. Nielsen

K. L. Nielsen
Head, Mathematics Division

Approved by
W. A. Key
W. A. Key
Head, Research and Test Department

TABLE OF CONTENTS

Foreword i

1. INTRODUCTION 1

2. THE FUNDAMENTAL FORMULA 1

3. EQUALLY SPACED INTERVALS 5

4. INTERPOLATION 10

   A. Equally Spaced Intervals 10

   B. Unequally Spaced Intervals 11

   C. Inverse Interpolation 14

   D. Multiple Interpolation 15

5. DIFFERENTIATION 17

   A. The Derivative at x_k for Unequally Spaced Intervals 18

   B. The Derivative at x_k for Equally Spaced Intervals. 21

6. INTEGRATION 26

7. SOLUTION OF DIFFERENTIAL EQUATIONS 32

8. HIGHER DERIVATIVES 36

9. LAGRANGIAN MULTIPLIERS 37

10. ACCURACY OF LAGRANGIAN FORMULAE 40

TABLES

I.	Stirling Numbers of the First Kind	41
II.	Lagrangian Interpolation Coefficients	42-43
III.	Differentiating Coefficients - Fractional Form	44
IV.	Differentiating Coefficients - Decimal Form.	45
V.	Integrating Coefficients	46-47
VI.	Second Derivative Coefficients	48
VII.	Third Derivative Coefficients	49

1. INTRODUCTION

One of the most applicable formulas in numerical analysis is one which passes an n^{th} degree polynomial through $(n + 1)$ given points. This formula is credited to Lagrange and is usually referred to as Lagrange's interpolation formula since its primary use was for interpolation problems. However, it can be used as the basis for many other numerical analysis problems and in special cases tables of coefficients may be calculated which reduce the problem to simply one of obtaining the sums of products. It is the purpose of this paper to present the complete development of the formulae and list some special tables.

2. THE FUNDAMENTAL FORMULA

Let there be given the values of the ordinates $y_o, y_1, \ldots, y_n$ of the function $y = f(x)$ at the $(n + 1)$ points $x_o, x_1, \ldots, x_n$. The polynomial of the n^{th} degree through these points can be written in the form

$$(1) \qquad L(x) = \frac{(x - x_1)(x - x_2) \ldots (x - x_n)}{(x_o - x_1)(x_o - x_2) \ldots (x_o - x_n)} y_o$$

$$+ \frac{(x - x_o)(x - x_2) \ldots (x - x_n)}{(x_1 - x_o)(x_1 - x_2) \ldots (x_1 - x_n)} y_1$$

$$+ \ldots\ldots\ldots\ldots$$

$$+ \frac{(x - x_o) \ldots (x - x_{i-1})(x - x_{i+1}) \ldots (x - x_n)}{(x_i - x_o) \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots (x_i - x_n)} y_i$$

$$+ \ldots\ldots\ldots\ldots$$

$$+ \frac{(x - x_o)(x - x_1) \ldots (x - x_{n-1})}{(x_n - x_o)(x_n - x_1) \ldots (x_n - x_{n-1})} y_n \, .$$

If we let

$$(2) \qquad P(x) = (x - x_o)(x - x_1) \cdots (x - x_n) = \prod_{j=o}^{n} (x - x_j)$$

and

$$(3) \qquad P_i(x) = (x - x_i)^{-1} P(x) = (x - x_i)^{-1} \prod_{j=o}^{n} (x - x_j)$$

then formula (1) becomes

$$(4) \qquad L(x) = \frac{P_o(x)}{P_o(x_o)} y_o + \frac{P_1(x)}{P_1(x_1)} y_1 + \cdots + \frac{P_n(x)}{P_n(x_n)} y_n .$$

It is easily seen that

$$P_r(x_i) = 0 \qquad \text{if} \qquad i \neq r$$

so that

$$(5) \qquad L(x_r) = \frac{P_r(x_r)}{P_r(x_r)} y_r = y_r \qquad (r = o, \ldots, n)$$

and the polynomial given by (1) is one which passes through the $(n + 1)$ points (x_r, y_r), $(r = o, \ldots, n)$. In general, $P_i(x)$ is a polynomial of degree n; i.e.,

$$(6) \qquad P_i(x) = a_{i,n} x^n + a_{i,n-1} x^{n-1} + \cdots + a_{i,o} \qquad \text{with} \qquad a_{in} = 1$$

and

$$(7) \qquad P_i(x_i) = a_{i,n} x_i^n + a_{i,n-1} x_i^{n-1} + \cdots + a_{i,o} = k_i .$$

It is desired that $L(x)$ approximate the function $f(x)$; i.e.,

(8) $$f(x) = L(x) + R(x)$$

where the remainder $R(x)$ is such that

(9) $$R(x_i) = 0 \qquad \text{for} \qquad (i = 0,1,\ldots,n)$$

Furthermore, if we let

$$R(x) = P(x)\, Q(x)$$

and consider any function

(10) $$\varphi(z) = f(z) - L(z) - P(z)\, Q(z)$$

such that

$$\varphi(x_i) = 0 \qquad \text{and} \qquad \varphi(x) = 0$$

if $x \neq x_i$ for all $i = 0,1,\ldots,n$; then $\varphi(z)$ vanishes at $n + 2$ points. By repeated application of Rolle's theorem $\varphi^{n+1}(\xi)$ vanishes where $x_0 \leq \xi \leq x_n$. However,

$$L^{n+1}(z) = 0 \qquad \text{and} \qquad P^{n+1}(z) = (n + 1)!$$

so that upon differentiating equation (10) we have

$$0 = f^{n+1}(\xi) - (n + 1)!\, Q(x)$$

or

$$Q(x) = \frac{1}{(n+1)!} f^{n+1}(\xi)$$

and at $z = x$ equation (10) yields

(11) $$f(x) = L(x) + P(x) f^{n+1}(\xi)/(n+1)!$$

Thus we may write

(12) $$f(x) = \sum_{i=o}^{n} \frac{P_i(x)}{P_i(x_i)} y_i + \frac{P(x)}{(n+1)!} f^{n+1}(\xi)$$

where ξ lies in the interval $[x_o \ldots x_n]$.

The last term of formula (12) is in a sense a remainder term and is a measure of the accuracy of the fit of the polynomial $L(x)$.

Formula (4) is invariant under a linear transformation. Let us make the transformation

$$x = hu + a \qquad x_i = hu_i + a$$

then

$$P_i(x) = (x - x_i)^{-1} P(x) = (x - x_i)^{-1} (x - x_o)(x - x_1) \ldots (x - x_n)$$

$$= (hu + a - hu_i - a)^{-1} (hu + a - hu_o - a)(hu + a - hu_1 - a)\ldots(hu + a - hu_n - a)$$

$$= h^{-1}(u - u_i)^{-1} h^n (u - u_o)(u - u_1) \ldots (u - u_n)$$

$$= h^{n-1}(u - u_i)^{-1} P(u)$$

$$= h^{n-1} P_i(u)$$

$$P_i(x_i) = (x_i - x_o)(x_i - x_1) \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots (x_i - x_n)$$

$$= (hu_i+a-hu_o-a)\ldots(hu_i+a-hu_{i-1}-a)(hu_i+a-hu_{i+1}-a)\ldots(hu_i+a-hu_n-a)$$

$$= h^{n-1}(u_i - u_o)(u_i - u_1) \ldots (u_i - u_{i-1})(u_i - u_{i+1}) \ldots (u_i - u_n)$$

$$= h^{n-1} P_i(u_i)$$

Therefore

$$L(x) = \sum_{i=o}^{n} \frac{P_i(x)}{P_i(x_i)} y_i = \sum_{i=o}^{n} \frac{P_i(u)}{P_i(u_i)} y_i = L(u)$$

3. EQUALLY SPACED INTERVALS

If the values of x_i $(i = 0,\ldots,n)$ are given at equally spaced intervals, we have

(13) $$x_i = x_o + ih$$

where

$$h = \Delta x = x_1 - x_o = x_r - x_{r-1}, \quad (r = 1,\ldots,n)$$

and also let

(14) $$u = \frac{x - x_o}{h} \quad \text{or} \quad x = x_o + uh \ .$$

The polynomial $P_i(x)$ now takes a special form

$$(15) \qquad P_i(x) = (x - x_i)^{-1} \prod_{j=o}^{n} (x - x_j)$$

$$= (x_o + uh - x_o - ih)^{-1} \prod_{j=o}^{n} (x_o + uh - x_o - jh)$$

$$= h^{-1}(u - i)^{-1} \prod_{j=o}^{n} h(u - j)$$

$$= h^{n}(u - i)^{-1} \prod_{j=o}^{n} (u - j)$$

Now

$$(16) \qquad \prod_{j=o}^{n} (u - j) = u(u - 1) \ldots (u - n)$$

$$= S_o^{n+1} u^{n+1} + S_1^{n+1} u^{n} + \ldots + S_n^{n+1} u$$

where

S_i^{n+1} $(i = 0,\ldots,n)$ are Stirling's Numbers of the first kind.

These numbers have the property that

$$(17) \qquad S_i^{n+1} = S_i^{n} - n\, S_{i-1}^{n}$$

and this recurrence formula is easily used to build up a table of these numbers (see Table I).

For equally spaced intervals we have

(18) $$P_i(x_i) = \prod_{j=0}^{n} (x_i - x_j)_{j \neq i} = h^n \prod_{j=0}^{n} (i - j)_{j \neq i}$$

$$= h^n \, i!(n-i)! \, (-1)^{n-i}$$

so that

(19) $$\frac{P_i(x)}{P_i(x_i)} = \frac{h^n \prod_{j=0}^{n} (u-j)_{j \neq i}}{h^n \, i!(n-i)!} \quad (-1)^{n-i}$$

$$= (-1)^{n-i} [i!(n-i)!]^{-1} \prod_{j=0}^{n} (u-j)_{j \neq i}$$

and

(20) $$L(x) = L_o(u)y_o + L_1(u)y_1 + \ldots + L_n(u)y_n$$

where $L_i(u)$ is given by (19).

Since $L_i(u)$ are functions of n and u it is possible to compute tables of these coefficients and thus the value of $L(x)$ for a specified x can be computed by a sum of products. In order to cut down the range of values on u for such a table, a shift to a "center" value is made which changes the notation somewhat. Let us remember the $n + 1$ point $x_o, x_1, \ldots, x_n$ in the following manner.

Case 1. Even number of points: $n + 1 = 2r$.

In this case we could remember the points

$$x_{-r}, x_{-r+1}, \ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots, x_r$$

and we are faced with a choice of picking the "center" value to be either x_{-1} or x_1. The general practice is to pick x_{-1} to be the "center" point and call it x_o. The numbering then becomes

$$x_{-r+1},\ x_{-r+2},\ \ldots,\ x_{-1},\ x_0,\ x_1,\ x_2,\ \ldots,\ x_r\ .$$

Example. Given 6 points, they are numbered

$$x_{-2},\ x_{-1},\ x_0,\ x_1,\ x_2,\ x_3\ .$$

We now have

$$x_i = x_o + ih \qquad (i = -r+1,\ \ldots,\ -1,\ 1,\ \ldots,\ r)$$

and

$$x = x_o + ph$$

so that

$$\text{(21)} \qquad P_i(x) = (x - x_i)^{-1} \prod_{j=o}^{n} (x - x_j)$$

$$= h^{-1}\ (p - i)^{-1} \prod_{j=-r+1}^{r} (x_o + ph - x_o - jh)$$

$$= h^{2r-1}\ (p - i)^{-1} \prod_{j=-r+1}^{r} (p - j)\ .$$

The product

$$\prod_{j=-r+1}^{r} (p-j) = (p+r-1)(p+r-2)\ldots(p+1)p\ (p-1)\ldots(p-r).$$

$$\text{(22)} \qquad P_i(x_i) = \prod_{j=o}^{n} (x_i - x_j)_{j\neq i} = h^{2r-1} \prod_{j=r+1}^{r} (x_o + ih - x_o - jh) = h^{2r-1} \prod_{j=-r+1}^{r} (i-j)_{i\neq j}\ ,$$

$$= h^{2r-1}[(i+r-1)(i+r-2)\ldots(i+1)(i)\ldots(1)(-1)(-2)\ldots(r-i)]\ ,$$

$$= h^{2r-1}[(i+r-1)!\ (r-i)!][-1]^{r-i}$$

and the coefficients of y_i become

$$(23) \qquad A_i(p) = (-1)^{r-i}[(i+r-1)!(r-i)!]^{-1} \prod_{j=-r+1}^{r} (p-j)_{j\neq i} \quad .$$

In the tabulation of these coefficients the following property is useful

$$(24) \qquad A_i(p) = A_{1-i}(1-p) \quad .$$

<u>Case 2.</u> Odd number of points; $n + 1 = 2r + 1$

In this case the points are numbered

$$x_{-r},\ x_{-r+1},\ \ldots,\ x_{-1},\ x_0,\ x_1,\ \ldots,\ x_r$$

and

$$(25) \qquad A_i(p) = (-1)^{r-i}[(i+r)!(r-i)!]^{-1} \prod_{j=-r}^{r} (p-j)_{j\neq i}$$

with the property

$$(26) \qquad A_i(p) = A_{-i}(-p) \quad .$$

Let us consider as an example the case of 5 points, and let $p = .7$. It is required to find A_{-2}, A_{-1}, A_0, A_1, A_2. Substituting into formula (25) we obtain

$$A_{-2}(.7) = (-1)^{2+2}[(-2+2)!(2+2)!]^{-1}(.7+1)(.7+0)(.7-1)(.7-2)$$

$$= (-1)^4[(4)!]^{-1}\ [(1.7)(.7)(-.3)(-1.3)]$$

$$= \frac{1}{24}(.4641)$$

$$= .0193375$$

$$A_{-1}(.7) = (-1)^{2+1}[(-1+2)!(2+1)!]^{-1}[(.7+2)(.7+0)(.7-1)(.7-2)]$$

$$= -(3!)^{-1}[(2.7)(.7)(-.3)(-1.3)]$$

$$= -.12285.$$

$$A_0(.7) = (-1)^2[(2!)(2!)]^{-1}[(.7+2)(.7+1)(.7-1)(.7-2)]$$

$$= .447525.$$

$$A_1(.7) = (-1)^1[(3!)(1!)]^{-1}(.7+2)(.7+1)(.7+0)(.7-2)$$

$$= .69615.$$

$$A_2(.7) = (-1)^0[4!(0!)]^{-1}(.7+2)(.7+1)(.7+0)(.7-1)$$

$$= -.0401625.$$

Extensive tables of these coefficients have been published (see reference (a)). A limited table for five points is given in Table II.

4. INTERPOLATION

One of the prime applications of Lagrange's formula is in interpolation. The problem is to find a value of a function at a point x which falls between tabulated values at points x_0, x_1, ...

A. EQUALLY SPACED INTERVALS

The simplest case is the one in which the tabulated values are given at equally spaced intervals; then formula (20) gives the desired results. Furthermore, the coefficients may be tabulated and the problem is reduced to simply one of finding the sum of products of two numbers and is easily performed as one operation on a calculating machine. For this purpose the coefficients are tabulated about a "midpoint" and we have

(27) $f(x) = A_{-r}y_{-r} + A_{-r+1}y_{-r+1} + \cdots + A_0 y_0 + A_1 y_1 + \cdots + A_r y_r$.

The procedure is explained by an example.

Example 1.

Find $f(1.77)$ by a 5-point Lagrangian formula from the table of values

x_i	1.5	1.6	1.7	1.8	1.9	2.0
y_i	48.09375	65.53600	87.69705	115.47360	149.86915	192.00000
$A_i(.7)$	.0193375	-.1228500	.4475250	.6961500	-.0401625	
$A_i(-3)$		-.0261625	.2541500	.8895250	-.1368500	.0193375

Solution Let $x_0 = 1.7$. then $p = \dfrac{x - x_0}{h} = \dfrac{1.77 - 1.7}{.1} = .7$

The Lagrangian coefficients $A_i(.7)$ are obtained from Table II and listed above.

$$f(1.77) = \sum_{i=-2}^{2} A_i(.7)y_i = 106.49336$$

If we had let $x_0 = 1.8$, $p = \dfrac{1.77 - 1.8}{.1} = -.3$ we obtain

$$f(1.77) = \sum_{i=-2}^{2} A_i(-.3)y_i = 106.49348$$

B. UNEQUALLY SPACED INTERVALS

The case of unequally spaced intervals makes the method more difficult to handle but increases its importance since fewer methods are now available. It now becomes necessary to return to formula (4). A schematic may be devised which greatly aids the computation. First, a linear transformation is made on the given x_i, $(i = 0, \ldots, n)$ in order to obtain small integers in so far as is possible.

Form now the square array

$$(28)\quad \left\{\begin{array}{ccccc} x - x_o & x_o - x_1 & x_o - x_2 & \cdots & x_o - x_n \\ x_1 - x_o & x - x_1 & x_1 - x_2 & \cdots & x_1 - x_n \\ x_2 - x_o & x_2 - x_1 & x - x_2 & \cdots & x_2 - x_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n - x_o & x_n - x_1 & x_n - x_2 & \cdots & x - x_n \end{array}\right.$$

We note that the product of the principal diagonal is

$$(29)\qquad P(x) = (x - x_o)(x - x_1) \dots (x - x_n) \ .$$

The products of the elements in each row yield

$$(30)\qquad R_i = (x_i - x_o) \dots (x - x_i) \dots (x_i - x_n) = (x - x_i)\, P_i(x_i)$$

Thus

$$(31)\qquad \frac{P_i(x)}{P_i(x_i)} = \frac{(x - x_i)^{-1} P(x)}{(x - x_i)^{-1}\, R_i} = \frac{P(x)}{R_i}$$

and

$$(32)\qquad y = L(x) = \frac{P(x)}{R_o}\, y_o + \frac{P(x)}{R_1}\, y_1 + \frac{P(x)}{R_2}\, y_2 + \dots \frac{P(x)}{R_n}\, y_n$$

$$= P(x)\left[\frac{y_o}{R_o} + \frac{y_1}{R_1} + \dots \frac{y_n}{R_n}\right] .$$

The square array (28) may now be augmented by two columns R_i and y_i/R_i (for speedy operation R_i need not be recorded). The sum of the last column multiplied by the product of the terms of the main diagonal yields the desired result.

Example 2.

Given the values

x	0	3	9	12	15	21	27
y	150	108	0	-54	-100	-144	-84

Find y at $x = 18$.

Solution

We first make the transformation $x = 3s$, then

s_i	0	1	3	4	5	7	9

and $s = 6$. The array of $s - s_i$ and $s_i - s_j$ becomes after augmented by R_i and y_i/R_i

s_i-s_0	s_i-s_1	s_i-s_2	s_i-s_3	s_i-s_4	s_i-s_5	s_i-s_6	R_i	y_i/R_i
6	-1	-3	-4	-5	-7	-9	22680	.006613757
1	5	-2	-3	-4	-6	-8	-5760	-.01875
3	2	3	-1	-2	-4	-6	864	0
4	3	1	2	-1	-3	-5	-360	.150
5	4	2	1	1	-2	-4	320	-.3125
7	6	4	3	2	-1	-2	2016	-.071428571
9	8	6	5	4	2	-3	-51840	.001620370

The product of terms in principal diagonal, $P(s) = 540$. $\quad -.244444444 = \Sigma\, y_i/R_i$

$$y = (540)(-.244444444) = -132.$$

C. INVERSE INTERPOLATION

Lagrange's formula adapts itself very nicely to inverse interpolation since formula (2) is simply a relation between two variables, either of which may be considered the independent variable. Thus, we can write x as a function of y

$$(33) \qquad L(y) = \frac{P_o(y)}{P_o(y_o)} x_o + \frac{P_1(y)}{P_1(y_1)} x_1 + \cdots + \frac{P_n(y)}{P_n(y_n)} x_n \ .$$

Normally, the values of y_i will be unequally spaced and the method just described must in general be employed. Furthermore, it will be more difficult to find a linear transformation which will reduce the size of the numbers. Let us consider an example.

Example 3.

From a five-place table of natural sines we have

x	30	31	33	34	36
y	.50000	.51504	.54464	.55919	.58779

Find x when y = .52992.

Solution

Let y = .1s , then

s_i	5.0000	5.1504	5.4464	5.5919	5.8779	5.2992

and the computational form is

$s_i - s_o$	$s_i - s_1$	$s_i - s_2$	$s_i - s_3$	$s_i - s_4$	R_i	x_i/R_i
.2992	-.1504	-.4464	-.5919	-.8779	.010438234	2874.049
.1504	.1488	-.2960	-.4415	-.7275	-.002127679	-14569.867
.4464	.2960	-.1472	-.1455	-.4315	-.001221146	-27023.796
.5919	.4415	.1455	-.2927	-.2860	.003182957	10681.891
.8779	.7275	.4315	.2860	-.5787	-.045611921	-789.267
					Sum =	-28826.990

P(x) = -.001110065; x = 31.9998.

To five place accuracy x = 32.

D. MULTIPLE INTERPOLATION

Interpolation in tables of multiple arguments is a laborious procedure and is usually accomplished by repeated application of interpolation on a single variable. Repeated application of Lagrange's Formula can best be illustrated by example. We shall limit our attention to equally spaced intervals; for unequally spaced intervals repeated application of the technique of 4 B must be employed.

Consider $y = f(x,r)$ and suppose the table of values is arranged as follows

x/r	r_0	r_1	r_2	r_3	r_4	r_5
x_0	y_{00}	y_{01}	y_{02}	y_{03}	y_{04}	y_{05}
x_1	y_{10}	y_{11}	y_{12}	y_{13}	y_{14}	y_{15}
x_2	y_{20}	y_{21}	y_{22}	y_{23}	y_{24}	y_{25}
x_3	y_{30}	y_{31}	y_{32}	y_{33}	y_{34}	y_{35}
x_4	y_{40}	y_{41}	y_{42}	y_{43}	y_{44}	y_{45}
x_5	y_{50}	y_{51}	y_{52}	y_{53}	y_{54}	y_{55}

Let it be desired to find a value of $y(x,r)$ where $x_2 < x < x_3$ and $r_1 < r < r_2$ by a three-point Lagrangian formula.

We obtain $p_x = \dfrac{x - x_2}{\Delta x}$ and $p_r = \dfrac{r - r_1}{\Delta r}$ and utilize the points x_1, x_2, x_3 and r_0, r_1, r_2, denoting the Lagrangian coefficient by A^x_{-1}, A^x_0, A^x_1 and A^r_{-1}, A^r_0, A^r_1. If we interpolate first with respect to x at each r_i $(i = 0,1,2)$ we have

$$(34)\quad \begin{cases} y_{x0} = A^x_{-1}\, y_{10} + A^x_0\, y_{20} + A^x_1\, y_{30} \\ y_{x1} = A^x_{-1}\, y_{11} + A^x_0\, y_{21} + A^x_1\, y_{31} \\ y_{x2} = A^x_{-1}\, y_{12} + A^x_0\, y_{22} + A^x_1\, y_{32} \end{cases} .$$

Then interpolating with respect to r yields

$$(35)\quad \begin{aligned} y(x,r) &= A^r_{-1} y_{x0} + A^r_0\, y_{x1} + A^r_1\, y_{x2} \\ &= A^r_{-1}\, [A^x_{-1}\, y_{10} + A^x_0\, y_{20} + A^x_1\, y_{30}] \\ &\quad + A^r_0\, [A^x_{-1}\, y_{11} + A^x_0\, y_{21} + A^x_1\, y_{31}] \\ &\quad + A^r_1\, [A^x_{-1}\, y_{12} + A^x_0\, y_{22} + A^x_1\, y_{32}] \\ &= A^r_{-1}\, A^x_{1}\, y_{10} + A^r_{-1}\, A^x_0\, y_{20} + A^r_{-1}\, A^x_1\, y_{30} \\ &\quad + A^r_0\, A^x_{-1}\, y_{11} + A^r_0\, A^x_0\, y_{21} + A^r_0\, A^x_1\, y_{31} \\ &\quad + A^r_1\, A^x_{-1}\, y_{12} + A^r_1\, A^x_0\, y_{22} + A^r_1\, A^x_1\, y_{32} . \end{aligned}$$

It is seen that nothing is gained by attempting to use the last expression since this would require the recording of the 9 products $A^r_i\, A^x_j$ $(i,j = -1,0,1)$ while using the first expression only 4 recordings are necessary. The second method would have application only in the special case where p_x and p_r remain constant for a large number of interpolations. The most expeditious method, therefore, is to compute the three y_{xi} $(i = 0,1,2)$ and then

$$(36)\quad y(x,r) = A^r_{-1}\, y_{x0} + A^r_0\, y_{x1} + A^r_1\, y_{x2} .$$

5. DIFFERENTIATION

Lagrange's formula may be used to find the derivative of a function $y = f(x)$ which is known only at discrete values x_i $(i = 0, \ldots, n)$. From formula (8) we have

$$f(x) = L(x) + R(x)$$

so that

(37) $$\frac{dy}{dx} = f'(x) = L'(x) + R'(x) .$$

Since by formula (11)

$$R(x) = P(x)\, f^{(n+1)}(\xi)/(n+1)!$$

(38) $$R'(x) = P'(x)\, f^{(n+1)}(\xi)/(n+1)! + P(x)\, f^{(n+2)}(\xi)/(n+1)!$$

The second term of this expression would be difficult to find even if it is known that $f^{(n+2)}(x)$ exists. However, if it were evaluated at a given point, x_i, then $P(x_i) = 0$ and only the first term remains. In practice the remainder term is only used to check the accuracy.

The derivative of $L'(x)$ is given by

(39) $$L'(x) = L_o'(x)y_o + L_1'(x)y_1 + \ldots + L_n'(x)y_n$$

and

(40) $$L_i'(x) = \frac{1}{P_i(x_i)} \frac{d}{dx} P_i(x)$$

$$= \frac{1}{P_i(x_i)} [(x-x_o) \ldots (x-x_{i-1})(x-x_{i+1}) \ldots (x-x_n)]$$

$$\left[\frac{1}{x-x_o} + \ldots + \frac{1}{x-x_{i-1}} + \frac{1}{x-x_{i+1}} + \ldots + \frac{1}{x-x_n}\right]$$

$$(40) \qquad L_i'(x) = \frac{1}{P_i(x_i)} \left[\prod_{j=o}^{n} (x - x_j)_{j \neq i} \right] \left[\sum_{j=o}^{n} \frac{1}{x - x_j} \right]_{j \neq i}$$

$$= \frac{P_i(x)}{P_i(x_i)} \left[\sum_{j=o}^{n} \frac{1}{x - x_j} \right]_{j \neq i} .$$

Formula (39) gives an expression for finding the derivative of a function at a general value of x. The computation of the coefficients of y_i $(i = 0, \ldots, n)$; namely, $L_i'(x)$, as given by formula (40) is somewhat difficult to manage. It can best be handled by forming the square array (28) and computing $P_i(x)/P_i(x_i)$ as was done by formula (31).

A. THE DERIVATIVE AT x_k FOR UNEQUALLY SPACED INTERVALS

Of special interest is the derivative evaluated at one of the given points x_k $(i = 0, \ldots, n)$. Formula (40) takes on two forms:

<u>(a) $k \neq i$.</u> In this case the product $\prod_{j=o}^{n} (x - x_j)_{j \neq i} = 0$ except when it is multiplied by $(x_k - x_k)^{-1}$ thus

$$(41) \qquad L_i'(x) = \frac{1}{P_i(x_i)} \left[\prod_{j=o}^{n} (x_k - x_j)_{\substack{j \neq i \\ j \neq k}} \right]$$

$$= \frac{1}{P_i(x_i)} [(x_k - x_o) \ldots (x_k - x_{k-1})(x_k - x_{k+1}) \ldots (x_k - x_{i-1}) \ldots (x_k - x_n)]$$

$$= \frac{1}{P_i(x_i)} [(x_k - x_o) \ldots (x_k - x_{k-1})(x_k - x_{k+1}) \ldots (x_k - x_n)]/(x_k - x_i)$$

$$= \frac{1}{P_i(x_i)} \prod_{j=o}^{n} (x_k - x_j)_{j \neq k} (x_k - x_i)^{-1}$$

If we let

(42) $$D_{ik} = (x_k - x_i)\, P_i(x_i)\,, \qquad i \neq k$$

we have

(43) $$L_i'(x) = \frac{1}{D_{ik}} \prod_{j=0}^{n} (x_k - x_j)_{j \neq k} \,.$$

(b) $k = i$. In this case we have

(44) $$\pi(x_i - x_j)_{j \neq i} = P_i(x_i)$$

and

(45) $$L_i'(x) = \frac{P_i(x_i)}{P_i(x_i)} \sum_{\substack{j=0 \\ j \neq i}}^{n} \frac{1}{x_i - x_j}$$

$$= \sum_{j=0, \neq i}^{n} (x_i - x_j)^{-1} \,.$$

Thus the complete expression for $L'(x_k)$ becomes

(46) $$L'(x_k) = L_0'(x_k)y_0 + \dots + L_k'(x_k)y_k + \dots + L_n'(x_k)y_n$$

$$= \frac{y_0}{D_{0k}} \prod_{j=0}^{n} (x_k - x_j)_{j \neq k} + \frac{y_1}{D_{1k}} \prod_{j=0}^{n} (x_k - x_j)_{j \neq k} + \dots$$

$$+ y_k \sum_{j=0}^{n} (x_k - x_j)^{-1}_{j \neq k} + \dots + \frac{y_n}{D_{nk}} \prod_{j=0}^{n} (x_k - x_j)_{j \neq k}$$

$$= \prod_{j=0}^{n} (x_k - x_j)_{j \neq k} \left[\frac{y_0}{D_{0k}} + \frac{y_1}{D_{1k}} + \dots + \frac{y_n}{D_{nk}} \right] + y_k \sum_{j=0}^{n} (x_k - x_j)^{-1}_{j \neq k}$$

or

$$(47) \qquad L'(x) = \prod_{j=0}^{n} (x_k - x_j)_{j \neq k} \left[\sum_{i=0}^{n} y_i \, D_{ik}^{-1} \right]_{i \neq k} + y_k \sum_{j=0}^{n} (x_k - x_j)^{-1}_{j \neq k} \; .$$

This formula for the derivative of $y = f(x)$ at $x = x_k$ can be found by the use of a schematic similar to the one given in Section 4.B. Form the square array

$$\left| \begin{array}{ccccccc}
 & x_0 - x_1 & x_0 - x_2 & \cdots & x_0 - x_k & \cdots & x_0 - x_n \\
x_1 - x_0 & & x_1 - x_2 & \cdots & x_1 - x_k & \cdots & x_1 - x_n \\
x_2 - x_0 & x_2 - x_1 & & \cdots & x_2 - x_k & \cdots & x_2 - x_n \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
x_k - x_0 & x_k - x_1 & x_k - x_2 & \cdots & & \cdots & x_k - x_n \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
x_n - x_0 & x_n - x_1 & x_n - x_2 & \cdots & x_n - x_k & \cdots &
\end{array} \right|$$

Notice that no element appears on the main diagonal. Now the product of the elements of each row is $P_i(x_i)$. If this be again multiplied by $(x_i - x_k)$ we have $-D_{ik}$. Thus we can augment this array by two columns, D_{ik} and $y_i \, D_{ik}^{-1}$; the first being obtained by multiplying the product of each row by the element in the k^{th} column (i.e., this element twice in the product) and changing sign. There will be no entry in the $(i = k)$ row. The product $\prod_{j=0}^{n} (x_k - x_j)_{j \neq k} = \prod_{j=0}^{n} [-(x_j - x_k)]_{j \neq k}$ is obtained by multiplying together the negative of the elements in the k^{th} column.

The elements of the second augmented column are now summed and multiplied by this product to yield the first term of the formula (47).

The second term is obtained by summing the negative reciprocals of the terms in the k^{th} column as can be easily seen from its expression. The final value of the derivative is then given by formula (47). Let us illustrate the procedure with an example.

Example 4.

Let us consider the example 2 given in 4.B, and let it be desired to obtain the derivative at $s_i = 5$; i.e., $x = 12$.

$s_i - s_o$	$s_i - s_1$	$s_i - s_2$	$s_i - s_3$	$s_i - s_4$	$s_i - s_5$	$s_i - s_6$	D_{i4}	$y_i\ D_{i4}^{-1}$
	-1	-3	-4	-5	-7	-9	18900	.007936507
1		-2	-3	-4	-6	-8	-4608	-.023437500
3	2		-1	-2	-4	-6	576	0
4	3	1		-1	-3	-5	-180	.300000000
5	4	2	1		-2	-4		
7	6	4	3	2		-2	4032	-.035714285
9	8	6	5	4	2		-69120	.001215278
$\pi(s_4 - s_i) = 320.$								.250

$$\Sigma(s_4 - s_i)^{-1} = \frac{1}{5} + \frac{1}{4} + \frac{1}{2} + \frac{1}{1} - \frac{1}{2} - \frac{1}{4} = \frac{1}{5} + 1 = \frac{6}{5} = 1.2000$$

$$L'(x) = (320)(.25) + (1.2)(-100) = -40.$$

B. THE DERIVATIVE AT x_k FOR EQUALLY SPACED INTERVALS

If the values of the independent variable are given at equally spaced intervals, we have from equation (20)

$$(48) \qquad L(x) = L_o(u)y_o + L_1(u)y_1 + \dots + L_n(u)y_n$$

where

$$(49) \qquad L_i(u) = (-1)^{n-i}\,[i!(n-i)!]^{-1} \prod_{j=o}^{n} (u-j)_{j\neq i} \; .$$

Since $x = uh + x_o$ we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{h}\frac{dy}{du}$$

and

(50) $$y' \doteq L'(x) = \frac{1}{h}\sum_{i=o}^{n} L_i'(u)y_i$$

and

(51) $$L_i'(u) = (-1)^{n-i}[i!(n-i)!]^{-1}\frac{d}{du}\prod_{j=o}^{n}(u-j)_{j\neq i} .$$

Furthermore [see formula (40)]

(52) $$\frac{d}{du}\prod_{j=o}^{n}(u-j)_{j\neq i} = \left[\prod_{j=o}^{n}(u-j)_{j\neq i}\right]\left[\sum_{j=o}^{n}(u-j)^{-1}_{j\neq i}\right] .$$

When $x = x_k$ we have

(53) $$u = \frac{x - x_o}{h} = \frac{x_k - x_o}{h} = k$$

so that at this point formula (52) takes on two special forms:

<u>(a) $k \neq i$</u>

(54) $$\frac{d}{du}\prod_{j=o}^{n}(u-j)_{j\neq i} = [u(u-1)\ldots(u-k+1)(u-k-1)\ldots(u-i+1)(u-i-1)\ldots(u-n)]$$

$$= k(k-1)\ \ldots\ 1\ (-1)\ldots(k-i+1)(k-i-1)\ldots(k-n)$$

$$= (k-i)^{-1}\ k!\ (n-k)!\ (-1)^{n-k}$$

and

$$(55) \qquad L_i'(u) = (-1)^{n-i}[i!(n-i)!]^{-1}(k-i)^{-1}k!(n-k)!(-1)^{n-k}$$

$$= (-1)^{i+k} \frac{k!(n-k)!}{i!(n-i)!(k-i)}$$

Note: $(-1)^{2n-i-k} = (-1)^{2n}(-1)^{-(i+k)} = (-1)^{i+k}$

(b) $k = i$

$$(56) \qquad \frac{d}{du} \prod_{j=o}^{n} (u-j)_{j \neq i} = [i(i-1)\dots(2)(1)(-1)(-2)\dots(i-n)]$$

$$\left[\frac{1}{i} + \frac{1}{i-1} + \dots \frac{1}{1} - \frac{1}{1} + \dots + \frac{1}{i-n}\right]$$

$$= i!(n-i)!(-1)^{n-i} \sum_{j=o}^{n} \frac{1}{i-j}\Bigg|_{j \neq i}$$

so that

$$(57) \qquad L_i'(u) = (-1)^{n-i} \frac{i!(n-i)!(-1)^{n-i}}{i!(n-i)!} \left[\sum_{j=o}^{n} \frac{1}{i-j}\right]_{j \neq i}$$

$$= \frac{1}{i} + \frac{1}{i-1} + \dots + \frac{1}{2} + \frac{1}{1} - \frac{1}{1} - \frac{1}{2} - \dots - \frac{1}{n-i}$$

$$= \sum_{j=1}^{i} \frac{1}{j} - \sum_{j=1}^{n-i} \frac{1}{j}$$

$$= - \sum_{j=i+1}^{n-i} \frac{1}{j} \qquad \text{if} \qquad 2i \leq n-1 ,$$

$$= \sum_{j=n-i+1}^{i} \frac{1}{j} \qquad \text{if} \qquad 2i > n-1 .$$

Thus the coefficients of y_i $(i = 0, \ldots, n)$ are all functions of n, k, and i, can be computed once and for all, and tabulated. Furthermore, the table of coefficients is "negatively symmetric" about the "midpoint" so that it is necessary to tabulate only half of them. Let the coefficients be denoted by A'_{ki}, then for equally spaced intervals

(58) $$y'(x_k) = L'(x_k) = \frac{1}{h}\,[A'_{ko}\,y_o + A'_{ki}\,y_i + \ldots + A'_{kn}\,y_n] \quad ,$$

and A'_{ki} are given by formulas (55) and (57).

Now for $k > m$ where $m = \frac{1}{2}n + 1$ if n is even and $m = \frac{1}{2}(n+1)$ if n is odd, we have

(59) $$A'_{ki} = -A'_{n-k,n-i} \quad .$$

By formula (55) we have for $k \neq i$

$$A'_{ki} = (-1)^{i+k}\,\frac{k!\,(n-k)!}{i!\,(n-i)!\,(k-i)}$$

and

$$A'_{n-k,n-i} = (-1)^{(n-k)+(n-i)}\,\frac{(n-k)!\;(n-[n-k])!}{(n-i)!\,(n-[n-i])!\;(n-k-[n-i])}$$

$$= (-1)^{k+i}\,\frac{(n-k)!\;(k!)}{(n-i)!\;(i)!\;(i-k)}$$

$$= (-1)^{k+i-1}\,\frac{(n-k)!\;k!}{(n-i)!\;(i)!\;(k-i)} \quad ,$$

thus

$$A'_{ki} = -A'_{n-k,n-i} \quad .$$

For $k = i$ we have by formula (57)

$$A'_{kk} = \sum_{j=n-k+1}^{k} \frac{1}{j} \qquad \text{and} \qquad A'_{n-k,n-k} = -\sum_{j=n-k+1}^{n-n+k} \frac{1}{j}$$

so that

$$A'_{kk} = -A'_{n-k,n-k} \ .$$

The tabulation of A'_{ki} can be made in two ways. The first is to obtain the lowest common denominator for the A'_{ki} for each n and tabulate the coefficients of y_i as integers with a final division to obtain the derivative. Thus

$$y'(x_k) = \frac{1}{h\,D_n} \sum_{i=0}^{n} a'_{ki}\, y_i \tag{60}$$

where $a'_{ki} = A'_{ki}\, D_n$

and D_n is the lowest common denominator of A'_{ki}. Table III gives the values of D_n and a'_{ki} for 10 points. The second method is to publish A'_{ki} in decimal form, see Table IV.

Example 5.

Find $\frac{dy}{dx}$ at $x = 1.7$ in the data given in Example 1.

Here $u = \frac{1.7 - 1.5}{.1} = 2 = k$; $n = 5$ $h = .1$

From Tables III ($D_n = 60$) and IV, we have

y_i	48.09375	65.53600	87.69705	115.47360	149.86915	192.00000
a_{2i}	3	-30	-20	60	-15	2
A_{2i}	.050000	-.500000	-.333333	1.000000	-.250000	.033333

$$y_2' = 10 \left[\sum_{i=0}^{5} a_{2i}' \; y_i \right] / 60 = 248.10650.$$

$$y_2' = 10 \left[\sum_{i=0}^{5} A_{2i}' \; y_i \right] = 248.10652.$$

6. INTEGRATION

As in the case of differentiation Lagrange's formula may also be used to find the integral of a function $f(x)$ which is known only at discrete values x_i $(i = 0,\ldots,n)$. That is the integral of $L(x)$ will be a good approximation of the integral of $f(x)$ so that we may write

$$\int_a^b f(x)dx = \alpha_0 y_0 + \alpha_1 y_1 + \cdots + \alpha_n y_n + R \tag{61}$$

where

$$\alpha_i = \frac{1}{P_i(x_i)} \int_a^b P_i(x)dx \qquad (i = 0,\ldots,n) \tag{62}$$

and

$$R = \frac{1}{(m+1)!} \int_a^b f^{(m+1)}(\xi)\, P(x)\, dx . \tag{63}$$

The problem of finding the integral of $f(x)$ then reduces to the evaluation of the coefficients α_i and the sum of products. The coefficients α_i depend upon the $n + 1$ points, x_i, and the limits of integration (a,b). If we write $P_i(x)$ as a polynomial in descending powers of x

$$(64) \qquad P_i(x) = x^n + a_{i,n-1}x^{n-1} + \dots + a_{i,1}x + a_{i,o}$$

the integral may be written in the form

$$(65) \qquad \int P_i(x)dx = b_{i,n+1}x^{n+1} + b_{i,n}x^n + \dots + b_{i,1}x + b_{i,o} \, .$$

where

$$(66) \qquad \begin{cases} b_{i,n+1} = \dfrac{1}{n+1} & (i = 0,\dots,n) \\ b_{i,j} = \dfrac{1}{j}\,(a_{i,j-1}) & (i = 0,\dots,n;\ j = 1,\dots,n) \\ b_{io} = \text{constants of integration.} & \end{cases}$$

The coefficients b_{ij} may be placed in a tabular form and augmented by a row of the constants $P_i(x_i)$. It is not necessary to write b_{io} since they will disappear when the limits of integration are inserted. Thus

x^{n+1}	x^n	x^{n-1}	...	x^2	x	$P_i(x_i)$
$b_{o,n+1}$	$b_{o,n}$	$b_{o,n-1}$		$b_{o,2}$	$b_{o,1}$	$P_o(x_o)$
...	...	...		...	...	...
$b_{n,n+1}$	$b_{n,n}$	$b_{n,n-1}$		$b_{n,2}$	$b_{n,1}$	$P_n(x_n)$

There remains now to consider the range of integration. If this range changes so that it is desired to have many integrals, the values for a_i may be tabulated in the following manner.

i	$x_0 - x_1$	$x_0 - x_2$	...	$x_1 - x_5$	...	$x_3 - x_8$	...
0	α_0	α_0	...	α_0	...	α_0	...
1	α_1	α_1	...	α_1	...	α_1	...
2	α_2	α_2	...	α_2	...	α_2	...
⋮	⋮	⋮	...	⋮	...	⋮	...
n	α_n	α_n	...	α_n	...	α_n	...

For the case of equally spaced intervals

$$P_i(x) = P_i(u) = u(u-1)\ldots(u-i+1)(u-i-1)\ldots(u-n) = \sum_{j=0}^{n} a_{i,j}\, u^j \tag{67}$$

and since $\quad dx = h\, du$

$$\int P_i(x)\, dx = h \int P_i(u)\, du\,. \tag{68}$$

The values of a_{ij} may be found easily by use of Stirling's numbers of the first kind. Consider for example the case of 4 points

$$\begin{cases} P_0(u) = (u-1)(u-2)(u-3) = u^3 - 6u^2 + 11u - 6\,, \\ P_1(u) = u(u-2)(u-3) = u^3 - 5u^2 + 6u\,, \\ P_2(u) = u(u-1)(u-3) = u^3 - 4u^2 + 3u\,, \\ P_3(u) = u(u-1)(u-2) = u^3 - 3u^2 + 2u\,. \end{cases}$$

If we note that

$$P_i(u) = P(u)/(u - i)$$

the polynomial $P(u)$ has for its coefficients the Stirling numbers of the first kind and from which we remove the factor $(u - i)$. This can be done by a synthetic division and easily put into a schematic. The removal of $(u - o)$ does not change the numbers so that the first polynomial has the Stirling numbers of the first kind as coefficients.

	u^3	u^2	u	u^0	Division
a_{oj}	1	-6	11	-6	0
		1	-5		
a_{1j}	1	-5	6		1
		2	-8		
a_{2j}	1	-4	3		2
		+3	-9		
a_{3j}	1	-3	2		3

The coefficients b_{ij} are now easily written down

	u^4	u^3	u^2	u	$P_i(u_i)$	L.C.D.
b_{oj}	$\frac{1}{4}$	$-\frac{6}{3}$	$\frac{11}{2}$	$-\frac{6}{1}$	-6	12
b_{1j}	$\frac{1}{4}$	$-\frac{5}{3}$	$\frac{6}{2}$		2	12
b_{2j}	$\frac{1}{4}$	$-\frac{4}{3}$	$\frac{3}{2}$		-2	12
b_{3j}	$\frac{1}{4}$	$-\frac{3}{3}$	$\frac{2}{2}$		6	12

$P_i(u_i)$ are easily found by the formula

$$P_i(u_i) = i!(n-i)!\ (-1)^{n-i}$$

and realizing that $P_o(u_o) = a_{oo}$ and then they alternate in sign. A column for the lowest common denominator (L.C.D.) is attached in case it is desired to factor out the denominator; it is to be recalled that $P_i(u_i)$ is a denominator.

Let us now consider a specific integral, say

$$\int_o^2 f(x)\,dx$$

for 4 points. We have

$$\alpha_o = \frac{h}{P_o(u_o)} \int_o^2 P(u)$$

$$= \frac{h}{P_o(u_o)} [b_{04}u^4 + b_{03}u^3 + b_{02}u^2 + b_{01}u]_o^2$$

$$= \frac{h}{-6} \left[\frac{1}{4}(2)^4 - \frac{6}{3}(2)^3 + \frac{11}{2}(2)^2 - 6\ (2)\right]$$

$$= -\frac{h}{6}\left[\frac{1}{12}\right] [(3)(2)^4 - 24(2)^3 + 66(2)^2 - 72(2)]$$

$$= -\frac{h}{72} [48 - 192 + 264 - 144]$$

$$= -\frac{h}{72} (-24)$$

$$= \frac{2}{6} h\ .$$

The schematic eases the computation

	$2^4 = 16$	$2^3 = 8$	$2^2 = 4$	2	L.C.D.	$P_i(u_i)$	$\alpha_i h^{-1}$
b_{0j}	3	-24	66	-72	12	-6	$\frac{1}{3}$
b_{1j}	3	-20	36		12	2	$\frac{4}{3}$
b_{2j}	3	-16	18		12	-2	$+\frac{1}{3}$
b_{3j}	3	-12	12		12	6	0

$$\int_0^2 f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2 + 0\ y_3]$$

For equally spaced intervals the values of α_i can be tabulated for the range of integration 0 - 1, 0 - 2, etc. The value of an integral between other integral limits can be found by recalling that

$$\int_2^4 f(x)dx = \int_0^4 f(x)dx - \int_0^2 f(x)dx \ .$$

Values of α_i are given in Table V and their use can be illustrated by an example.

Example 6:

Find the $\int_{1.5}^{1.8} y\ dx$ given the table

x_i	1.5	1.6	1.7	1.8	1.9	2.0
y_i	48.09375	65.53600	87.69705	115.47360	149.86915	192.

Solution

$$\int_{1.5}^{1.8} f(x)dx = h \int_{0}^{3} f(u)du$$

$$= h \left[\sum_{i=0}^{5} \alpha_i (0\text{-}3)\, y_i \right]$$

From Table V for six points in the interval 0-3 we have

α_i	.31875	1.36875	.71250	.71250	-.13125	.01875

and since $h = .1$

$$\int_{1.5}^{1.8} y dx = .1\ (233.721045) = 23.372104$$

7. Solution of Differential Equations

The methods most frequently used in the numerical solution of differential equations are those called step-by-step methods in which the values of the dependent variable are calculated from a sequence of equally spaced values of the independent variable. That is, if we have a differential equation

(69) $$y' = f(x,y)$$

and the initial conditions x_o, y_o, we compute a sequence of values for y, say y_1, y_2, y_3, ..., for equally spaced chosen values of x, say x_1, x_2, x_3, ... To do this we rely upon formulas which yield the next value of y, say y_{i+1}, from previously known values of y and its derivative. Such recursion formulas for y_{i+1} may be derived from the Lagrangian formula for finding the derivative of a function.

Let us apply formula (58) to the case of five points, we have (See Table III)

$$
(70)\quad \begin{cases}
y_0' = \dfrac{1}{12h}\left(-25y_0 + 48y_1 - 36y_2 + 16y_3 - 3y_4\right) \\[2ex]
y_1' = \dfrac{1}{12h}\left(-3y_0 - 10y_1 + 18y_2 - 6y_3 + y_4\right) \\[2ex]
y_2' = \dfrac{1}{12h}\left(y_0 - 8y_1 + 8y_3 - y_4\right) \\[2ex]
y_3' = \dfrac{1}{12h}\left(-y_0 + 6y_1 - 18y_2 + 10y_3 + 3y_4\right) \\[2ex]
y_4' = \dfrac{1}{12h}\left(3y_0 - 16y_1 + 36y_2 - 48y_3 + 25y_4\right)
\end{cases}
$$

If we consider the formulas for y_1', y_2', and y_3' we can solve for y_4 in terms of these derivatives and y_0

$$
(71)\quad \begin{cases}
12h\, y_1' = -3y_0 - 10y_1 + 18y_2 - 6y_3 + y_4 \\
12h\, y_2' = y_0 - 8y_1 + 8y_3 - y_4 \\
12h\, y_3' = -y_0 + 6y_1 - 18y_2 + 10y_3 + 3y_4
\end{cases}
$$

We have from the second equation of (71)

$$
(72)\qquad y_4 = y_0 - 8y_1 + 8y_3 - 12h\, y_2'
$$

Adding the first and third equation of (71) gives

$$
12h\,(y_1' + y_3') = -4y_0 - 4y_1 + 4y_3 + 4y_4
$$

or

$$
y_1 - y_3 = -y_0 - 3h(y_1' + y_3') + y_4
$$

and upon substitution of this into equation (72) we get

$$
y_4 = y_0 - 8\left[-y_0 - 3h(y_1' + y_3') + y_4\right] - 12h\, y_2'
$$

or

$$
9y_4 = 9y_0 + 12h(2y_1' - y_2' + 2y_3')
$$

so that

$$(73) \qquad y_4 = y_0 + \frac{4h}{3} (2y_1' - y_2' + 2y_3').$$

This formula yields a value of y in terms of the value of y four steps previous and the values of the derivative of y at the preceeding three steps. This is the well known "predictor" formula in Milne's method for the solution of differential equations. With this "predictor" or extrapolation formula there is used a "corrector" formula

$$(74) \qquad y_{n+1} = y_{n-1} + \frac{h}{3} (y_{n+1}' + 4y_n' + y_{n-1}').$$

which may be obtained from the last three equations of (70) by simply multiplying y_3' by 4 and adding:

$$\begin{array}{rcl}
y_2' & = & \frac{1}{12h} [\; y_0 - 8y_1 \qquad + 8y_3 - y_4 \;] \\
4y_3' & = & \frac{1}{12h} [-4y_0 + 24y_1 - 72y_2 + 40y_3 + 12y_4] \\
y_4' & = & \frac{1}{12h} [\; 3y_0 - 16y_1 + 36y_2 - 48y_3 + 25y_4] \\
\hline
y_2' + 4y_3' + y_4' & = & \frac{1}{12h} [-36y_2 + 36y_4].
\end{array}$$

Although formulas (73) and (74) will keep the solution going once it has been started it will not start the solution. The problem may be started by using Taylor's Series expansion but more frequently it is started by successive approximations using the following formulas

$$(75) \qquad \left\{ \begin{array}{l}
y_1 = y_0 + \frac{h}{24} (7y_1' + 16y_0' + y_{-1}') + \frac{h^2 \, y_0''}{4} \\
y_{-1} = y_0 - \frac{h}{24} (y_1' + 16y_0' + 7y_{-1}') + \frac{h^2 \, y_0''}{4} \\
y_2 = y_0 + \frac{2h}{3} (5y_1' - y_0' - y_{-1}') - 2h^2 y_0'' \\
y_2 = y_0 + \frac{h}{3} (y_2' + 4y_1' + y_0')
\end{array} \right.$$

which may be obtained by algebraic manipulation of the Lagrangian derivative formulas. Trial values to start the method of successive approximation are given by the simple Euler formulas

$$(76.) \quad \begin{cases} y_1' = y_0' + hy_0'' \\ y_{-1}' = y_0' - hy_0'' \end{cases}$$

The procedure will be illustrated by an example.

Example 7. Solve the differential equation

$$\frac{dy}{dx} = 2x - y$$

if $x_0 = 1, y_0 = 3$.

Solution Differentiating y' we get

$$y'' = 2 - y' = 2 - 2x + y$$

and evaluation at (x_0, y_0) yields $y_0'' = 3$.

The calculation may be arranged as follows

$$y' = 2x - y$$

1. The Start from Formulas (75) and (76)

y_{-1}	3.115	3.1155	3.115512	3.115512
y'_{-1}	-1.3	-1.315	-1.3155	-1.315512
y_1	2.915	2.9145	2.914513	2.914513
y'_1	-.7	-.715	-.7145	-.714513
y_2			2.856197	2.856192
y'_2			-.456197	-.456192

$h/24 = .004166667$

$y_0'' = 3.$

$h^2 y_0'' / 4 = .0075$

2. The Continuation from (73) and (74)

x	y_c	y'_c	y_p	y'_p
.9	3.115512	-1.315512		
1.0	3.	-1.		
1.1	2.914513	-.714513		
1.2	2.856192	-.456192		
1.3	2.822456	-.222456	2.822463	-.222463
1.4	2.810960	-.010960	2.810968	-.010968
1.5	2.819593	.180407	2.819600	.180400
1.6	2.846434	.353566	2.846440	.353560
1.7	2.889756	.510244	2.889763	.510237
1.8	2.947985	.652015	2.947991	.652009
1.9	3.019708	.780292	3.019715	.780285
2.0	3.103635	.896365	3.103641	.896359

$h/3 = .033333$

$4h/3 = .133333$

8. Higher Derivatives

The higher derivatives are obtained by repeated differentiation of equation (39). Thus

(77) $$L''(x) = L_o''(x)\, y_o + L_1''(x)\, y_1 + \ldots + L_n''(x)\, y_n$$

where

(78) $$L_i''(x) = \frac{1}{P_i(x_i)} \frac{d^2}{dx^2} [\, P_i(x) \,]$$

Although a notation and schematic can be devised for the general case it is very unwieldy. Consequently, we shall develop only the special case of equally spaced intervals. From equation (50) we have

(79) $$y'(x) = \frac{1}{h} \sum_{i=o}^{n} L_i'(u) y_i$$

so that

(80) $$y''(x) = \frac{1}{h^2} \sum_{i=o}^{n} L_i''(u) y_i$$

Now

$$L_i(u) = (-1)^{n-i} \frac{n!}{i!\,(n-i)!} (u-i)^{-1} \binom{u}{n} .$$

where $\binom{u}{n}$ is the binomial coefficient notation.

Let

(81) $$b_i = \frac{(-1)^{n-i}}{i!\,(n-i)!}$$

then $$L_i'(u) = b_i \frac{d}{du} \left[\prod_{j=o}^{n} (u-j)_{j \neq i} \right] .$$

and

(82) $$L_i''(u) = b_i \frac{d^2}{du} \left[\prod_{j=o}^{n} (u-j)_{j \neq i} \right] .$$

These derivatives may now be evaluated at $u = 0, 1, 2, 3, \ldots, n$ to yield the values of the derivatives at $x = x_o, x_1, \ldots, x_n$. Thus we can write

(83) $$y''(x_k) = h''(x_k) = \frac{1}{h^2} [A_{ko}'' y_o + A_{k1}'' y_1 + A_{k2}'' y_2 + \ldots + A_{kn}'' y_n]$$

where

$$A_{ki}'' = L_i''(k) .$$

which may be computed once and for all and have been tabulated in Table VI.

The continuation to higher derivatives is immediate. The third derivative is given by

$$(84) \qquad y'''(x_k) = L'''(x_k) = \frac{1}{h^3}[A'''_{ko}y_o + A'''_{k1}y_1 + \cdots + A'''_{kn}y_n]$$

and the values for A'''_{ki} have been tabulated in Table VII.

Example 8.

Find the first, second, and third derivatives at $x = 1$ of the function whose values are

x	-2	-1	0	1	2	3	4
y	104	17	0	-1	8	69	272

Solution

Using Tables III, VI, VII with $n = 6$, $h = 1$, $x_3 = 1$, $u = 3$ we write

$$y'(1) = \frac{1}{60}[-1(104) + 9(17) - 45(0) + 0(-1) + 45(8) - 9(69) + 1(272)]$$

$$= \boxed{1} .$$

$$y''(1) = \frac{1}{180}[2(104) - 27(17) + 270(0) - 490(-1) + 270(8) - 27(69) + 2(272)]$$

$$= \boxed{6} .$$

$$y'''(1) = \frac{1}{24}[3(104) - 24(17) + 39(0) + 0(-1) - 39(8) + 24(69) - 3(272)]$$

$$= \boxed{18} .$$

9. Lagrangian Multipliers

Although the Lagrangian multiplier is not directly associated with the Lagrangian formulas discussed this far it still seems appropriate to include them in this paper because they are concerned with the finding of an extremum of a function which is constrained by some additional condition. In the principle of least squares which is so often used in numerical analysis we are concerned with the problem of minimizing the sum of the squares of the residuals of the analytical expression and the values found by substitution of the known points. If now an additional constraint is imposed upon the parameters in the analytical expression the problem could be solved by use of the Lagrangian multiplier.

Let us consider a function of two variables; $f(x,y)$ and let it be desired to find the maximum and minimum values of this function if furthermore x and y are related by $g(x,y) = 0$.

The necessary condition for an extremum is that the total differential vanishes. Thus for the function $f(x,y)$

$$(85) \qquad \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial g}\, dy = 0 .$$

The total differential of the constrained relation is

$$(86) \qquad \frac{\partial g}{\partial x}\, dx + \frac{\partial g}{\partial g}\, dy = 0 .$$

If now equation (86) is multiplied by an undetermined multiplier, λ, and added to equation (85) we have

$$(87) \qquad \left(\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y}\right) dy = 0 .$$

Equation (87) would be satisfied if λ were determined so that

$$(88) \qquad \begin{cases} \left(\dfrac{\partial f}{\partial x} + \lambda \dfrac{\partial g}{\partial x}\right) = 0 \\ \left(\dfrac{\partial f}{\partial y} + \lambda \dfrac{\partial g}{\partial y}\right) = 0 \\ g(x,y) = 0 \end{cases}$$

The multiplier λ is called the Lagrangian multiplier.

Let us apply this multiplier to a problem of fitting data by the principle of least squares. Consider the problem of fitting a set of data by the expression $y = mx + b$. The principle of least squares forms the n residuals

$$v_i = m x_i + b - y_i \qquad (i = 1, \ldots\ldots, n)$$

and minimizes the sum of the squares of these residuals

$$\Sigma v_i^2 = f(m,b)$$

$$= \Sigma (mx_i + b - y_i)^2 ,$$

$$\frac{\partial f}{\partial m} = 2m\Sigma x_i^2 + 2\Sigma x_i b - 2\Sigma x_i y_i ,$$

$$\frac{\partial f}{\partial b} = 2m\Sigma x_i + 2nb - 2\Sigma y_i .$$

If it is now further required that

$$g(m,b) = m + 2b = 0 ,$$

we have $\dfrac{\partial g}{\partial m} = 1$, $\dfrac{\partial g}{\partial b} = 2$,

and the three equations

$$\begin{cases} 2(\Sigma x_i^2 m + \Sigma x_i b - \Sigma x_i y_i) + \lambda = 0 \\ 2(\Sigma x_i m + nb - \Sigma y_i) + 2\lambda = 0 \\ m + 2b = 0 \end{cases}$$

to solve for m, b, and λ : Letting

$$S_1 = \Sigma x_i , \quad S_2 = \Sigma x_i^2 , \quad W_1 = \Sigma y_i , \quad W_2 = \Sigma x_i y_i$$

we may write

$$\begin{cases} 2S_2 m + 2S_1 b - 2W_2 + \lambda = 0 , \\ S_1 m + nb - W_1 + \lambda = 0 , \\ m + 2b = 0 ; \end{cases}$$

$$m = -2b ,$$

$$(-4S_2 + 2S_1)b + \lambda = 2W_2 ,$$

$$(n - 2S_1)b + \lambda = W_1 ,$$

$$\lambda = 2W_2 + (4S_2 - 2S_1)b ,$$

$$(n - 2S_1)b + 2W_2 + (4S_2 - 2S_1)b = W_1 ,$$

$$(n + 4S_2 - 4S_1)b = W_1 - 2W_2 ,$$

$$\begin{cases} b = \dfrac{W_1 - 2W_2}{n + 4S_2 - 4S_1} \\ m = -2b . \end{cases}$$

Example 9

Fit a straight line to the data such that the slope is a minus twice the intercept.

x	1	2	3	4	5	6	7
y	.4	1.1	1.5	2.2	2.6	3.1	3.8

Solution

$n = 7,\ S_1 = 28,\ W_1 = 14.7,\ W_2 = 74.1,\ S_2 = 140$

$$b = \frac{14.7 - 2(74.1)}{7 + 4(140) - 4(28)} = \frac{-133.5}{455} = -.2934$$

$$m = .5868$$

$$\boxed{y = .5868x - .2934}$$

Check:

x	1	2	3	4	5	6	7
y	.2934	.8802	1.467	2.0538	2.6406	3.2274	3.8142

This, of course, is not the best fitting straight line; it is the best fitting straight line for which $m = -2b$.

10. Accuracy of Lagrangian Formulae

All of the formulas derived in this paper are based on the assumption that the given data can be fitted by a polynomial in the region under discussion and this initial assumption should frequently be checked. The inherent error in the Lagrangian formulae is exhibited in the remainder term of equation (12)

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!} P(x)$$

where ξ is an arbitrary point in the interval $(0, n)$.

This remainder term may be carried throughout all of the derivatives of the formulas and is a measure of their accuracy. Thus in formula (58) for the derivative should be added the remainder

$$\frac{h^n}{c} y^{(n+1)} .$$

For the integrals we have the expression (63).

The accuracy should always be tested whenevor possible.

TABLE I.

Stirling Numbers of the First Kind

$$S_i^{(n+1)} = S_i^{(n)} - nS_{i-1}^{(n)}$$

n	$S_0^{(n)}$	$S_1^{(n)}$	$S_2^{(n)}$	$S_3^{(n)}$	$S_4^{(n)}$	$S_5^{(n)}$	$S_6^{(n)}$	$S_7^{(n)}$	$S_8^{(n)}$	$S_9^{(n)}$	$S_{10}^{(n)}$	$S_{11}^{(n)}$
1	1											
2	1	-1										
3	1	-3	2									
4	1	-6	11	-6								
5	1	-10	35	-50	24							
6	1	-15	85	-225	274	-120						
7	1	-21	175	-735	1624	-1764	720					
8	1	-28	322	-1960	6769	-13132	13068	-5040				
9	1	-36	546	-4536	22449	-67284	118124	-109584	40320			
10	1	-45	870	-9450	63273	-269325	723680	-1172700	1026576	-3628800		
11	1	-55	1320	-10320	157773	-902055	3415930	-8409500	12753575	-13894560	36288000	
12	1	-66	1925	-24840	271293	-2637558	13339535	-45995730	105258076	-154183896	189128160	399168000

TABLE II.

Lagrangian Interpolation Coefficients

5 - Points

p	A_{-2}	$-A_{-1}$	A_0	A_1	$-A_2$	p
0.00	.0000000	.0000000	1.0000000	.0000000	.0000000	0.00
.01	.0008291	.0065998	.9998750	.0067332	.0008374	.01
.02	.0016493	.0130654	.9995000	.0135986	.0016827	.02
.03	.0024603	.0193956	.9988752	.0205954	.0025352	.03
.04	.0032614	.0255898	.9980006	.0277222	.0033946	.04
0.05	.0040523	.0316469	.9968766	.0349781	.0042602	0.05
.06	.0048325	.0375662	.9955032	.0423618	.0051315	.06
.07	.0056016	.0433468	.9938810	.0498722	.0060079	.07
.08	.0063590	.0489882	.9920102	.0575078	.0068890	.08
.09	.0071045	.0544894	.9898914	.0652676	.0077740	.09
0.10	.0078375	.0598500	.9875250	.0731500	.0086625	0.10
.11	.0085577	.0650692	.9849116	.0811538	.0095538	.11
.12	.0092646	.0701466	.9820518	.0892774	.0104474	.12
.13	.0099580	.0750814	.9789464	.0975196	.0113425	.13
.14	.0106373	.0798734	.9755960	.1058786	.0122387	.14
0.15	.0113023	.0845219	.9720016	.1143531	.0131352	0.15
.16	.0119526	.0890266	.9681638	.1229414	.0140314	.16
.17	.0125879	.0933870	.9640838	.1316420	.0149266	.17
.18	.0132077	.0976030	.9597624	.1404530	.0158203	.18
.19	.0138119	.1016740	.9552008	.1493730	.0167116	.19
0.20	.0144000	.1056000	.9504000	.1584000	.0176000	0.20
.21	.0149718	.1093806	.9453612	.1675324	.0184847	.21
.22	.0155269	.1130153	.9400856	.1767682	.0193651	.22
.23	.0160652	.1165052	.9345746	.1861058	.0202403	.23
.24	.0165862	.1198490	.9288294	.1955430	.0211098	.24
0.25	.0170898	.1230469	.9228516	.2050781	.0219727	0.25
.26	.0175757	.1260990	.9166[illegible]	.2147090	.0228283	.26
.27	.0180437	.1290052	.9102036	.2244338	.0236758	.27
.28	.0184934	.1317658	.9035366	.2342502	.0245[illegible]46	.28
.29	.0189248	.1343806	.8966432	.2441564	.0253437	.29
0.30	.0193375	.1368500	.8895250	.2541500	.0261625	0.30
.31	.0197314	.1391740	.8821838	.2642290	.0269701	.31
.32	.0201062	.1413530	.8746214	.2743910	.0277658	.32
.33	.0204619	.1433870	.8668398	.2846340	.0285436	.33
.34	.0207981	.1452766	.8588408	.2949554	.0293179	.34
0.35	.0211148	.1470219	.8506266	.3053531	.0300727	.35
.36	.0214118	.1486234	.8421990	.3158246	.0308122	.36
.37	.0216890	.1500814	.8335604	.3263676	.0315355	.37
.38	.0219461	.1513966	.8247128	.3369794	.0322419	.38
.39	.0221832	.1525692	.8156586	.3476578	.0329303	.39
0.40	.0224000	.1536000	.8064000	.3584000	.0336000	0.40
.41	.0225965	.1544894	.7969394	.3692036	.0342500	.41
.42	.0227725	.1552382	.7872792	.3800658	.0348795	.42
.43	.0229281	.1558468	.7774220	.3909842	.0354874	.43
.44	.0230630	.1563162	.7673702	.4019558	.0360730	.44
0.45	.0231773	.1566469	.7571266	.4129781	.0366352	.45
.46	.0232709	.1568398	.7466936	.4240482	.0371731	.46
.47	.0233438	.1568956	.7360742	.4351634	.0376857	.47
.48	.0233958	.1568154	.7252710	.4463206	.0381722	.48
.49	.0234271	.1565998	.7142870	.4575172	.0386314	.49
0.50	.0234375	.1562500	.7031250	.4687500	.0390625	0.50
$-p$	A_2	$-A_1$	A_0	A_{-1}	$-A_{-2}$	$-p$

TABLE II.

Lagrangian Interpolation Coefficients

5 - Points

p	A_{-2}	$-A_{-1}$	A_0	A_1	$-A_2$	p
0.50	.0234375	.1562500	.7031250	.4687500	.0390625	0.50
.51	.0234271	.1557668	.6917380	.4800162	.0394644	.51
.52	.0233958	.1551514	.6602790	.4913126	.0398362	.52
.53	.0233438	.1544046	.6686012	.5026364	.0401767	.53
.54	.0232709	.1535278	.6567576	.5139842	.0404851	.54
0.55	.0231773	.1525219	.6447516	.5253531	.0407602	0.55
.56	.0230630	.1513882	.6325862	.5367398	.0410010	.56
.57	.0229281	.1501278	.6202650	.5481412	.0412064	.57
.58	.0227725	.1487422	.6077912	.5595538	.0413755	.58
.59	.0225965	.1472324	.5951684	.5709746	.0415070	.59
0.60	.0224000	.1456000	.5824000	.5824000	.0416000	0.60
.61	.0221832	.1438462	.5694896	.5938268	.0416533	.61
.62	.0219461	.1419726	.5564408	.6052514	.0416659	.62
.63	.0216890	.1399804	.5432574	.6166706	.0416363	.63
.64	.0214118	.1378714	.5299430	.6280806	.0415642	.64
0.65	.0211148	.1356469	.5165016	.6394781	.0414477	0.65
.66	.0207981	.1333086	.5029368	.6508594	.0412859	.66
.67	.0204619	.1308580	.4892528	.6622210	.0410776	.67
.68	.0201062	.1282970	.4754534	.6735590	.0408218	.68
.69	.0197314	.1256270	.4615428	.6848700	.0405171	.69
0.70	.0193375	.1228500	.4475250	.6961500	.0401625	0.70
.71	.0189248	.1199676	.4334042	.7073954	.0397567	.71
.72	.0184934	.1169818	.4191846	.7186022	.0392986	.72
.73	.0180437	.1138942	.4048706	.7297668	.0387868	.73
.74	.0175757	.1107070	.3904664	.7408850	.0382203	.74
0.75	.0170898	.1074219	.3759766	.7519531	.0375977	0.75
.76	.0165862	.1040410	.3614054	.7629670	.0369178	.76
.77	.0160652	.1005662	.3467576	.7739228	.0361793	.77
.78	.0155269	.0969998	.3320370	.7848162	.0353811	.78
.79	.0149718	.0933436	.3172502	.7956434	.0345217	.79
0.80	.0144000	.0896000	.3024000	.8064000	.0336000	0.80
.81	.0138119	.0857710	.2874918	.8170820	.0326146	.81
.82	.0132077	.0818590	.2725304	.8276850	.0315643	.82
.83	.0125879	.0778660	.2575208	.8382050	.0304476	.83
.84	.0119526	.0737946	.2424678	.8486374	.0292634	.84
0.85	.0113023	.0696469	.2273766	.8589781	.0280102	0.85
.86	.0106373	.0654254	.2122520	.8692226	.0266867	.86
.87	.0099580	.0611324	.1970994	.8793666	.0252915	.87
.88	.0092646	.0567706	.1819238	.8894054	.0238234	.88
.89	.0085577	.0523422	.1667306	.8993348	.0222808	.89
0.90	.0078375	.0478500	.1515250	.9091500	.0206625	0.90
.91	.0071045	.0432964	.1363124	.9188466	.0189670	.91
.92	.0063590	.0386842	.1210982	.9284198	.0171930	.92
.93	.0056016	.0340158	.1058880	.9378652	.0153389	.93
.94	.0048325	.0292942	.0906872	.9471778	.0134035	.94
0.95	.0040523	.0245219	.0755016	.9563531	.0113852	0.95
.96	.0032614	.0197018	.0603366	.9653862	.0092826	.96
.97	.0024603	.0148366	.0451982	.9742724	.0070942	.97
.98	.0016493	.0099294	.0300920	.9830066	.0048187	.98
.99	.0008291	.0049828	.0150240	.9915842	.0024544	.99
1.00	.0000000	.0000000	.0000000	1.0000000	.0000000	1.00
-p	A_2	$-A_1$	A_0	A_{-1}	$-A_{-2}$	-p

TABLE III.

Differentiation Coefficients - Fractional Form

$$A_{ki} = \left. \frac{dA_i}{du} \right|_{u=k}$$

k \ i	A'_{k0}	A'_{k1}	A'_{k2}	A'_{k3}	A'_{k4}	A'_{k5}	A'_{k6}	A'_{k7}	A'_{k8}	A'_{k9}	k
					n = 2 (3 points) $D_n = 2$						
0	-3	4	-1								2
1	-1	0	1								
					n = 3 (4 points) $D_n = 6$						
0	-11	18	-9	2							3
1	-2	-3	6	-1							2
					n = 4 (5 points) $D_n = 12$						
0	[illegible]	48	-36	16	-3						4
1	-3	-10	18	-6	1						3
2	1	-8	0	8	-1						
					n = 5 (6 points) $D_n = 60$						
0	-137	300	-300	200	-75	12					5
1	-12	-65	120	-60	20	-3					4
2	3	-30	-20	60	-15	2					3
					n = 6 (7 points) $D_n = 60$						
0	-147	360	-450	400	-225	72	-10				6
1	-10	-77	150	-100	50	-15	2				5
2	2	-24	-35	80	-30	8	-1				4
3	-1	9	-45	0	45	-9	1				
					n = 7 (8 points) $D_n = 420$						
0	-1089	2940	-4410	4900	-3675	1764	-490	60			7
1	-60	-609	1260	-1050	700	-315	84	-10			6
2	10	-140	-329	700	-350	140	-35	4			5
3	-4	42	-252	-105	420	-42	28	-3			4
					n = 8 (9 points) $D_n = 840$						
0	-2283	6720	-11760	15680	-14700	9804	-3920	960	-105		8
1	-105	-1338	2940	-2940	2540	-1470	588	-140	15		7
2	15	-240	-798	1680	-1050	560	-210	48	-5		6
3	-5	60	-420	-378	1050	-420	140	-30	3		5
4	3	-32	168	-672	0	672	-168	32	-3		5
					n = 9 (10 points) $D_n = 2520$						
0	-7129	22680	-45360	70560	-79380	63504	-35280	12960	-2835	280	9
1	-280	-4329	10080	-11760	11760	-8820	4704	-1680	360	-35	8
2	35	-630	-2754	5880	-4410	2940	-1470	504	-105	10	7
3	-10	135	-1080	-1554	3780	-1890	840	-270	54	-5	6
4	5	-60	360	-1680	-504	2520	-840	240	-45	4	5
k	$-A_{k,n}$	$-A_{k,n-1}$	$-A_{k,n-2}$	$-A_{k,n-3}$	$-A_{k,n-4}$	$-A_{k,n-5}$	$-A_{k,n-6}$	$A_{k,n-7}$	$A_{k,n-8}$	$A_{k,n-9}$	k

TABLE IV.

Differentiation Coefficients - Decimal Form

$$A'_{ki} = \left.\frac{dA_i}{du}\right|_{u=k}$$

$$A'_{ki} = -A'_{n-k,n-i} \qquad k > \tfrac{1}{2}n$$

n = 2

k \ i	A'_{k0}	A'_{k1}	A'_{k2}
0	-1.5	2.0	-.5
1	-.5	0	.5

n = 3

k \ i	A'_{k0}	A'_{k1}	A'_{k2}	A'_{k3}
0	-1.833333	3.000000	-1.500000	.333333
1	-.333333	-.500000	1.000000	-.166667

n = 4

k \ i	A'_{k0}	A'_{k1}	A'_{k2}	A'_{k3}	A'_{k4}
0	-2.083333	4.000000	-3.000000	1.333333	-.250000
1	-.250000	-.833333	1.500000	-.500000	.083333
2	.083333	-.666667	0	.666667	-.083333

n = 5

k \ i	A'_{k0}	A'_{k1}	A'_{k2}	A'_{k3}	A'_{k4}	A'_{k5}
0	-2.283333	5.000000	-5.000000	3.333333	-1.250000	.200000
1	-.200000	-1.083333	2.000000	-1.000000	.333333	-.050000
2	.050000	-.500000	-.333333	1.000000	-.250000	.033333

n = 6

k \ i	A'_{k0}	A'_{k1}	A'_{k2}	A'_{k3}	A'_{k4}	A'_{k5}	A'_{k6}
0	-2.450000	6.000000	-7.500000	6.666667	-3.750000	1.200000	-.166667
1	-.166667	-1.283333	2.500000	-1.666667	.833333	-.250000	.033333
2	.033333	-.400000	-.583333	1.333333	-.500000	.133333	-.016667
3	-.016667	.150000	-.750000	0	.750000	-.150000	.016667

n = 7

k \ i	A'_{k0}	A'_{k1}	A'_{k2}	A'_{k3}	A'_{k4}	A'_{k5}	A'_{k6}	A'_{k7}
0	-2.592857	7.000000	-10.500000	11.666667	-8.750000	4.200000	-1.166667	.142857
1	-.142857	-1.450000	3.000000	-2.500000	1.666667	-.750000	.200000	-.023810
2	.023810	-.333333	-.783333	1.666667	-.833333	.333333	-.083333	.009524
3	-.009524	.100000	-.600000	-.250000	1.000000	-.100000	.066667	-.007143

n = 8

k \ i	A'_{k0}	A'_{k1}	A'_{k2}	A'_{k3}	A'_{k4}	A'_{k5}	A'_{k6}	A'_{k7}	A'_{k8}
0	-2.717857	8.000000	-14.000000	18.666667	-17.500000	11.200000	-4.666667	1.142857	-.125000
1	-.125000	-1.592857	3.500000	-3.500000	2.916667	-1.750000	.700000	-.166667	.017857
2	.017857	-.285714	-.950000	2.000000	-1.250000	.666667	-.250000	.057143	-.005952
3	-.005952	.071429	-.500000	-.450000	1.250000	-.500000	.166667	-.035714	.003571
4	.003571	-.038095	.200000	-.800000	0	.800000	-.200000	.038095	-.003571

n = 9

k \ i	A'_{k0}	A'_{k1}	A'_{k2}	A'_{k3}	A'_{k4}	A'_{k5}	A'_{k6}	A'_{k7}	A'_{k8}	A'_{k9}
0	-2.828968	9.000000	-18.000000	28.000000	-31.500000	25.200000	-14.000000	5.142857	-1.125000	.111111
1	-.111111	-1.717857	4.000000	-4.666667	4.666667	-3.500000	1.866667	-.666667	.142857	-.013889
2	.013889	-.250000	-1.092857	2.333333	-1.750000	1.166667	-.583333	.200000	-.041667	.003968
3	-.003968	.053571	-.428571	-.616667	1.500000	-.750000	.333333	-.107143	.021429	-.001984
4	.001984	-.023810	.142857	-.666667	-.200000	1.000000	-.333333	.095238	-.017857	.001587

TABLE V.

Integration Coefficients (α_i)

n = 2 (3 Points)

α_i	0 - 1	0 - 2
α_0	.416666667	.333333333
α_1	.666666667	1.333333333
α_2	-.083333333	.333333333

n = 3 (4 Points)

α_i	0 - 1	0 - 2	0 - 3
α_0	.375000000	.333333333	.375000000
α_1	.791666667	1.333333333	1.125000000
α_2	-.208333333	.333333333	1.125000000
α_3	.041666667	0	.375000000

n = 4 (5 Points)

α_i	0 - 1	0 - 2	0 - 3	0 - 4
α_0	.348611111	.322222222	.337500000	.311111111
α_1	.897222222	1.377777778	1.275000000	1.422222222
α_2	-.366666667	.266666667	.900000000	.533333333
α_3	.147222222	.044444444	.525000000	1.422222222
α_4	-.026388889	-.011111111	-.037500000	.311111111

n = 5 (6 Points)

α_i	0 - 1	0 - 2	0 - 3	0 - 4	0 - 5
α_0	.329861111	.311111111	.318750000	.311111111	.329861111
α_1	.990972222	1.433333333	1.368750000	1.422222222	1.302083333
α_2	-.554166667	.155555556	.712500000	.533333333	.868055556
α_3	.334722222	.155555556	.712500000	1.422222222	.868055556
α_4	-.120138889	-.066666667	-.131250000	.311111111	1.302083333
α_5	.018750000	.011111111	.018750000	0	.329861111

n = 6 (7 Points)

α_i	0 - 1	0 - 2	0 - 3	0 - 4	0 - 5	0 - 6
α_0	.315591931	.301322751	.305803571	.302645503	.307126323	.292857143
α_1	1.076587301	1.492063492	1.446428571	1.473015873	1.438492063	1.542857143
α_2	-.768204365	.008730159	.518303571	.406349206	.527033730	.192857143
α_3	.620105820	.351322751	.971428571	1.591534392	1.322751323	1.942857143
α_4	-.334176587	-.213492064	-.325446428	.184126984	.961061508	.192857143
α_5	.104365079	.069841270	.096428571	.050793651	.466269841	1.542857143
α_c	-.014269179	-.009788359	-.012946428	-.008465609	-.022734788	.292857143

n = 7 (8 Points)

α_i	0 - 1	0 - 2	0 - 3	0 - 4	0 - 5	0 - 6	0 - 7
α_0	.304224537	.292857143	.295758928	.294179894	.295758928	.292857143	.304224537
α_1	1.156159060	1.551322751	1.516741071	1.532275132	1.516063822	1.542857143	1.449016203
α_2	-1.006919642	-.169047619	.307366071	.228571428	.288318452	.192857143	.535937500
α_3	1.017964616	.647619047	1.322991071	1.887830687	1.720610119	1.942857143	1.210821759
α_4	-.732035383	-.509788359	-.677008928	-.112169312	.563202711	.192857143	1.210821759
α_5	.343080357	.247619048	.307366071	.228571428	.704985119	1.542857143	.535937500
α_6	-.093840939	-.069047619	-.083258928	-.067724867	-.102306547	.292857143	1.449016203
α_7	.011367394	.003465608	.010044643	.008465608	.011367394	0	.304224537

TABLE V. (Cont.)

n = 8 (9 Points)

a_i	0 - 1	0 - 2	0 - 3	0 - 4	0 - 5	0 - 6	0 - 7	0 - 8
a_0	.294868000	.285511423	.287522321	.286631393	.287319499	.286428571	.288439428	.279082892
a_1	1.231011353	1.610088153	1.582633928	1.592663139	1.585579254	1.594285714	1.575297067	1.661516754
a_2	-1.268902667	-.374726631	.075741071	.017213404	.052014440	.012857143	.093954475	-.261569485
a_3	1.541930665	1.058977072	1.754241071	2.310546737	2.193218143	2.302857143	2.094737506	2.961834215
a_4	-1.386992945	-1.023985890	-1.253571428	-.640564373	-.027557319	-.257142857	.105864197	-1.281128747
a_5	.867046406	.653977072	.751816071	.651287477	1.177593143	1.302857143	1.419903549	2.961334215
a_6	-.355323964	-.274726631	-.313313929	-.279082892	-.338010560	.112857143	1.007033179	-.261569486
a_7	.086219687	.067231040	.075937500	.068353615	.075582520	.051428571	.430505401	1.661516754
a_8	-.009356536	-.007345679	-.008236607	-.007548501	-.008439429	-.006428571	-.015785105	.279082892

n = 9 (10 Points)

a_i	0 - 1	0 - 2	0 - 3	0 - 4	0 - 5	0 - 6	0 - 7	0 - 8	0 - 9
a_0	.285975446	.279082892	.280546875	.280000000	.280344053	.280000000	.280546875	.279082892	.285975446
a_1	1.302044339	1.567945326	1.645412946	1.612345679	1.645356272	1.652142857	1.646330054	1.661516754	1.551127232
a_2	-1.553034611	-.606155202	-.174375000	-.221516755	-.199101631	-.215571429	-.190177455	-.261569485	.105482143
a_3	2.204905202	1.593977072	2.370171571	2.527583774	2.779155643	2.842857143	2.757762345	2.961834215	1.943035714
a_4	-2.381454750	-1.833985890	-2.132477679	-1.476119929	-.906463561	-1.067142857	-.883597505	-1.281128747	.580379464
a_5	1.861508212	1.463977072	1.647522321	1.498543033	2.056499332	2.712857143	2.414315384	2.961534215	.580379464
a_6	-1.018798500	-.814726631	-.899821428	-.836119929	-.924544051	-.427142857	.344055641	-.261569485	1.943035714
a_7	.370351631	.295659612	.327053571	.307583774	.329998395	.282857143	.714537345	1.561516754	.108482143
a_8	-.080389523	-.065202822	-.071015625	-.067231040	-.071213447	-.064285714	-.066815094	.279082592	1.551127232
a_9	.007892554	.006428571	.006975446	.006631393	.006975446	.006428571	.007892554	0	.285975446

n = 10 (11 Points)

a_i	0 - 1	0 - 2	0 - 3	0 - 4	0 - 5	0 - 6	0 - 7	0 - 8	0 - 9	0 - 10
a_0	.280189596	.273403746	.2745107[illegible]	.274153172	.274342755	.274153311	.274377905	.274020633	.275127333	.258341483
a_1	1.369902839	1.724736735	1.70577715[illegible]	1.710813959	1.705371120	1.710259740	1.705019723	1.712139383	1.599606360	1.775359414
a_2	-1.853397861	-.861716770	-.4460141[illegible]	-.434424017	-.469159445	-.460097402	-.467750375	-.459671315	-.424682934	-.810435705
a_3	3.019207201	2.280474587	3.0945495[illegible]	3.569203142	3.499309814	3.540259740	3.493033370	3.569305755	3.364809253	4.549462862
a_4	-3.206483247	-3.026506541	-3.400126826	-2.[illegible]953523	-2.166733367	-2.287597402	-2.154040650	-2.344203944	-1.907724228	-4.351551226
a_5	3.571542406	2.900121853	3.158701295	2.[illegible]243706	3.565523152	4.177402597	3.963945006	4.237524450	3.566103896	7.137646304
a_6	-2.443326997	-2.007347282	-2.157470575	-2.063953923	-2.147417559	-1.647597402	-.951424400	-1.324944634	-.545067978	-4.351551226
a_7	1.184653629	.980157120	1.551424512	1.009203142	1.050153068	.980259740	1.454913370	2.265938295	1.530255681	4.549462862
a_8	-.385752772	-.320764339	-.342654726	-.330333303	-.341276260	-.325811558	-.364421503	.051281064	1.047962155	-.810435705
a_9	.075751054	.063220031	.067339691	.065099674	.066958295	.064545455	.069552222	.050622628	.405456574	1.775359414
a_{10}	-.006785850	-.005579146	-.006035424	-.005348325	-.006001255	-.005511688	-.005168967	-.005062263	-.011348112	.258341483

TABLE VI.

Second Derivative Coefficients

$$A''_{ki} = \left. \frac{dA'_{ki}}{du} \right|_{u=k}$$

k	A''_{k0}	A''_{k1}	A''_{k2}	A''_{k3}	A''_{k4}	A''_{k5}	A''_{k6}	A''_{k7}

n = 2 (3 Points) D_n = 1

k	A''_{k0}	A''_{k1}	A''_{k2}
All	1	-2	1

n = 3 (4 Points) D_n = 1

k	A''_{k0}	A''_{k1}	A''_{k2}	A''_{k3}
0	2	-5	4	-1
1	1	-2	1	0
2	0	1	-2	1
3	-1	4	-5	2

n = 4 (5 Points) D_n = 12

k	A''_{k0}	A''_{k1}	A''_{k2}	A''_{k3}	A''_{k4}
0	35	-104	114	-56	11
1	11	-20	6	4	-1
2	-1	16	-30	16	-1
3	-1	4	6	-20	11
4	11	-56	114	-104	35

n = 5 (6 Points) D_n = 12

k	A''_{k0}	A''_{k1}	A''_{k2}	A''_{k3}	A''_{k4}	A''_{k5}
0	45	-154	214	-156	61	-10
[illegible]	10	-15	-4	14	-6	1
[illegible]	-1	16	-30	16	-1	0
[illegible]	0	-1	16	-30	16	-1
4	1	-6	14	-4	-15	10
5	-10	61	-156	214	-154	45

n = 6 (7 Points) D_n = 180

k	A''_{k0}	A''_{k1}	A''_{k2}	A''_{k3}	A''_{k4}	A''_{k5}	A''_{k6}
0	812	-3132	5265	-5080	2970	-972	137
1	137	-207	-255	470	-285	93	-13
2	-13	228	-420	200	15	-12	2
3	2	-27	270	-490	270	-27	2
4	2	-12	15	200	-420	228	-13
5	-13	93	-285	470	-255	-207	137
6	137	-972	2970	-5080	5265	-3132	812

n = 7 (8 Points) D_n = 180

k	A''_{k0}	A''_{k1}	A''_{k2}	A''_{k3}	A''_{k4}	A''_{k5}	A''_{k6}	A''_{k7}
0	938	-4014	7911	-9490	7380	-3618	1019	-126
1	126	-70	-486	855	-670	324	-90	11
2	-11	214	-378	130	85	-102	16	-2
3	2	-27	270	-490	270	-27	2	0
4	0	2	-27	270	-490	270	-27	2
5	-2	16	-102	85	130	-378	214	-11
6	11	-90	324	-670	855	-486	-70	126
7	-126	1019	-3618	7380	-9490	7911	-4014	938

TABLE VII

Third Derivative Coefficients

$$A'''_{ki} = \left. \frac{dA''_{ki}}{du} \right|_{u=k}$$

n = 3 (4 Points) D_n = 1

k	A'''_{k0}	A'''_{k1}	A'''_{k2}	A'''_{k3}	A'''_{k4}	A'''_{k5}	A'''_{k6}	A'''_{k7}
All	-1	3	-3	1				

n = 4 (5 Points) D_n = 2

k	A'''_{k0}	A'''_{k1}	A'''_{k2}	A'''_{k3}	A'''_{k4}	A'''_{k5}	A'''_{k6}	A'''_{k7}
0	-5	18	-24	14	-3			
1	-3	10	-12	6	-1			
2	-1	2	0	-2	1			
3	1	-6	12	-10	3			
4	3	-14	24	-18	5			

n = 5 (6 Points) D_n = 4

k	A'''_{k0}	A'''_{k1}	A'''_{k2}	A'''_{k3}	A'''_{k4}	A'''_{k5}	A'''_{k6}	A'''_{k7}
0	-17	71	-118	98	-41	7		
1	-7	25	-34	22	-7	1		
2	-1	-1	10	-14	7	-1		
3	1	-7	14	-10	1	1		
4	-1	7	-22	34	-25	7		
5	-7	41	-98	118	-71	17		

n = 6 (7 Points) D_n = 24

k	A'''_{k0}	A'''_{k1}	A'''_{k2}	A'''_{k3}	A'''_{k4}	A'''_{k5}	A'''_{k6}	A'''_{k7}
0	-147	696	-1383	1488	-921	312	-45	
1	-45	168	-249	192	-87	24	-3	
2	-3	-24	105	-144	87	-24	3	
3	3	-24	39	0	-39	24	-3	
4	-3	24	-87	144	-105	24	3	
5	3	-24	87	-192	249	-168	45	
6	45	-312	921	-1488	1383	-696	147	

n = 7 (8 Points) D_n = 120

k	A'''_{k0}	A'''_{k1}	A'''_{k2}	A'''_{k3}	A'''_{k4}	A'''_{k5}	A'''_{k6}	A'''_{k7}
0	-967	5104	-11787	15560	-12725	6432	-1849	232
1	-232	889	-1392	1205	-680	267	-64	7
2	-7	-176	693	-1000	715	-288	71	-8
3	8	-71	48	245	-440	267	-64	7
4	-7	64	-267	440	-245	-48	71	-8
5	8	-71	288	-715	1000	-693	176	7
6	-7	64	-267	680	-1205	1392	-889	232
7	-232	1849	-6432	12725	-15560	11787	-5104	967

REFERENCES

(a) Tables of Lagrangian Interpolation Coefficients, National Bureau of Standards, Columbia University Press. New York, 1944.

(b) Numerical Mathematical Analysis, J. B. Scarborough, Second Edition, The Johns Hopkins Press, 1950.

(c) Numerical Calculus, W. E. Milne. Princeton University Press, 1949.

(d) NOPI-MFF-2-48-D. Integration Formulas, K. L. Nielsen, 12 February 1948.

INITIAL DISTRIBUTION

Addresses:	Copies
Chief, Bureau of Ordnance, Navy Department, Washington 25, D.C. (Re4) - 1 (Re8e) - 1 (Re8) - 2 (Ad3) - 2 (Tech. Library) (Re8c) - 1 (Ad6) - 1 (Publications)	8
Chief, Bureau of Aeronautics, Navy Department, Washington, D.C. Attn: Armament Division, Rm 1W42	2
Chief of Naval Operations, Navy Department, Washington 25, D.C.	1
Chief, Office of Naval Research, Navy Department, Washington 25, D. C.	1
Commanding Officer, Naval Aviation Ordnance Test Station, Chincoteague, Virginia	1
Commander, Naval Ordnance Test Station, P. O. China Lake, Inyokern, California Attn: Aviation Ordnance Department	2
Director, Naval Research Laboratory, Washington, D.C.	1
Armed Services Technical Information Agency, Document Service Center, Knott Building, Dayton 2, Ohio	5
Superintendent, U.S. Naval Gun Factory (Code 724) Washington 25, D. C.	1
Commanding General, Building 313, Technical Information Section, Aberdeen Proving Ground, Maryland	2
Technical Information Section, Building 313, Development and Proof Services Aberdeen Proving Ground, Maryland	1
Commander, Naval Ordnance Laboratory, White Oak, Silver Spring 19, Maryland	2
Commander, Naval Air Test Center, Patuxent River, Maryland Attn: Armament Test Division	1
Commander, U. S. Naval Air Development Center, Johnsville, Pa. Attn: NADC Library	1
U. S. Naval Inspector of Ordnance, Applied Physics Laboratory The Johns Hopkins University, 8621 Georgia Avenue Silver Spring, Maryland	1
Operations Research Office, The Johns Hopkins University 6410 Connecticut Avenue, Chevy Chase, Maryland Attn: Library	1
Commanding General (WCLGF-1), Wright Air Development Center Wright-Patterson Air Force Base, Ohio	5
Commanding Officer & Director, Special Devices Center, Port Washington, New York Attn: Librarian	1
Commanding General, Redstone Arsenal, Huntsville, Alabama Attn: Technical Library	3
Office, Chief of Ordnance, Department of the Army Washington, D. C. Attn: ORDTB - 1 ORDTX-AR - 1	2

Addresses:	Copies
Commanding General, Air Force Armament Center, Eglin Air Force Base, Florida Attn: ACGL	1
Commanding General, Air Proving Ground, AFB, Eglin Field, Florida	1
Jet Propulsion Laboratory, California Institute of Technology, Pasadena 3, California Attn: Dr. L. G. Dunn	1
Dr. Walter Bartky, Institute for Air Weapons Research University of Chicago, Museum of Science and Industry Chicago 37, Illinois	1
The Franklin Institute Laboratories for Research and Development, 20th and Parkway, Philadelphia 3, Pa. Attn: Technical Librarian Electrical Engineering	1
The Rand Corporation, 1700 Main Street, Santa Monica, Calif.	2
Armour Research Foundation Technology Center, Chicago 16, Illinois	1
Purdue University, West Lafayette, Indiana	5